COLLECTIVE PROCESSES IN GRAVITATING SYSTEMS. I

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Translated from Astronomicheskii Zhurnal, Vol. 42, No. 4, pp. 709-717, July-August, 1965
Original article submitted September 13, 1964

Methods developed in plasma physics are used for the investigation of the dispersion equation of small oscillations in locally homogeneous segments of collisionless gravitating systems of particles. It is shown that such systems are unstable. A formula is derived for the calculation of the critical wavelength (in the sense of Jeans). Systems formed by two encountering interpenetrating "cold" and Maxwellian flows, and also "cold" and Maxwellian systems at rest, are considered. The instability is of an oscillatory character for wavelengths less than a certain λ_1, and thus differs from the aperiodic gravitational instability of a continuous medium. Application to the Galaxy shows that in the circumsolar neighborhood the time in which instability develops is of the order of $\tau \sim 2 \cdot 10^8$ years.

1. Introduction

Systems consisting of large numbers of interacting particles can be described by statistical methods. The simplest of such systems is a gas of neutral particles whose properties can be described by the Boltzmann equation for the distribution function. In connection with the development of plasma physics, it has been shown that the analogy between a plasma and an ordinary gas of neutral particles is not exact since a plasma is a self-consistent system. In such a system the mutual interaction between an individual particle and all others is governed by Coulomb's law, thus giving rise to peculiarities of the collective phenomena. The collective phenomena do not depend on the presence of ordinary collisions that determine the situation in atomic or molecular gases, and thus these phenomena also exist in a collisionless plasma. In cases where the plasma can be regarded as collisionless, the equations are simplified because of the absence of the collision integral in the Boltzmann equation [1].

Gravitating systems have certain similarities with collisionless plasma, even though there are significant differences. The study of collective phenomena in systems of self-gravitating particles, i.e., particles with gravitational interactions, by analogy with a collisionless plasma is of interest from the viewpoint of application to the dynamics of star clusters, the Galaxy, and agglomerations of galactic and related objects.

Collective phenomena in stellar dynamics have been treated by Sweet [2], for instance. According to his data, the number of stars in the "sphere of mutual interaction" is sufficiently large, and therefore the Boltzmann equation can be used. Since the stars in the Galaxy are surrounded by an interstellar gas, it is natural to expect a strong collective interaction of the stars with the gas. Sweet [2] in particular has proved that Jean's critical wavelength, at which instability of the gas sets in, can be decreased as a result of the relative velocity between star and gas. In the Galaxy, for instance, where the velocities of the Kapteyn flows are known, the interstellar gas is unstable at all wavelengths, i.e., Jean's critical wavelength is practically nonexistent. Thus, the attempt to calculate collective phenomena in the Galaxy leads to very interesting results. We are interested in another aspect of the question, namely the evolution of gravitational systems. We have therefore considered collective processes in stellar systems without taking account of the interstellar gas, since the influence of the latter on the motion of the stars is apparently not large.

Since the gravitational interaction is of opposite sign to the electrostatic interaction, results in the relation of small perturbations in gravitational systems should be qualitatively different from those in the case of collisionless plasma. Besides, unlike plasma, the equilibrium-state potential in the case of gravitational interaction is not neutralized. Thus, gravitational systems in nature are decidedly inhomogeneous. However, in small regions of the inhomogeneous systems it can be assumed that $\Phi_0 \approx$ const. In such regions one can apply the results obtained for homogeneous plasma. In this paper we give the results of a study of longitudinal waves in a locally homogeneous system using a linearized approximation.

The analogous problem for homogeneous collisionless plasma has been studied in numerous works. As is well known, it is necessary to clearly distinguish between two types of problems: (1) the problem of oscillations with prescribed initial and boundary conditions, and (2) the problem of oscillations caused by fluctuating local deviations of the distribution function from the equilibrium value. In problems of the first type, the solution must be strongly dependent on the initial and boundary conditions. The method of solution of such problems is that of integral transforms [3]. In the second case we can seek the solution in a form independent of initial conditions – for example, in the form of plane waves (substitutional analysis). Of course, this latter method does not yield exhaustive quantitative information. However, by such means one can obtain the dispersion equation, study the types of waves, and examine the question of stability.

Below, by analogy with an infinite homogeneous collisionless plasma, we will derive the dispersion equation for longitudinal waves in a locally homogeneous system of gravitating particles. Since we are interested in the evolution of the system, we will discuss its stability. We examined the following particular types of systems of gravitating particles: (1) a system of particles at rest, (2) a system of two streams of particles flowing with identical speeds in opposite directions, (3) a system of particles with Maxwellian distribution of velocities, (4) a system of two Maxwellian streams of particles with equal and opposite relative velocities. The type of the waves has been investigated in the third and fourth cases. We calculated increments in some boundary cases. They have been discussed for values of the parameters that are typical for the Galaxy.

2. Dispersion Equation

The basic equations of the problem are the Boltzmann equation for the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ without the collision integral and the Poisson equation for the gravitational potential $\Phi(\mathbf{r}, t)$. They form the system

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \nabla\Phi\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\Delta\Phi = -4\pi GM\int_{\mathbf{v}} f d\mathbf{v}, \tag{2}$$

where G is the gravitational constant, and M is the mass of the individual particle. We will consider particles of identical mass. However, as with a plasma, the results can easily be generalized to the case of an aggregate of different masses.

Let f experience a small disturbance $f_1(\mathbf{r}, \mathbf{v}, t)$ from the equilibrium value $f_0(\mathbf{v})$, i.e.,

$$f = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t), \tag{3}$$

Since the system is locally homogeneous, f_0 depends only on the velocity, and it has been normalized to N – the number of particles per cubic centimeter.

Correspondingly, Φ also suffers a small deviation $\varphi(\mathbf{r}, t)$ from the equilibrium value Φ_0, i.e.,

$$\Phi = \Phi_0 + \varphi(\mathbf{r}, t). \tag{4}$$

Substituting (3) and (4) into (1) and (2) and taking account of the smallness of f_1 and φ, we can obtain the following linearized system of equations for f_1 and φ:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} + \nabla\varphi\frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{5}$$

$$\Delta\varphi = -4\pi GM\int_{\mathbf{v}} f_1 d\mathbf{v}. \tag{6}$$

The solution for f_1 and φ will be sought in the form of the plane waves

$$f_1(\mathbf{r}, \mathbf{v}, t) = g(\mathbf{v})\exp[i(\mathbf{kr} - \omega t)], \tag{7}$$

$$\varphi(\mathbf{r}, t) = c\exp[i(\mathbf{kr} - \omega t)], \tag{8}$$

where $\mathbf{k}$ is the material wave vector, and ω is the complex frequency.

Substitution of (7) and (8) into (5) and (6) shows that system (5) and (6) has a solution of the form (7) and (8) provided the following relation is fulfilled:

$$k^2 = -\alpha\int\frac{\mathbf{k}\cdot\dfrac{\partial f_0}{\partial \mathbf{v}}}{\mathbf{kv} - \omega}d\mathbf{v}, \tag{9}$$

where $\alpha = 4\pi GM$. Relation (9) connects the frequency ω with the wave vector $\mathbf{k}$ and is thus the dispersion equation for such waves.

Since we are interested in longitudinal waves, the x axis will be chosen in the direction of the wave vector $\mathbf{k}$, and the integration in (9) will be carried out with respect to v_y and v_z. The integrated function $f_0(\mathbf{v})$ will again be denoted by f_0. Expression (9) thus leads to

$$k^2 = -\alpha \int_{-\infty}^{\infty} \frac{f_0' dv}{v-u}, \qquad (10)$$

where $u = \omega/k$, $k = |\mathbf{k}|$, $v = v_x$, and $f_0' = df_0/dv$.

In a plasma the corresponding dispersion equation has the form

$$k^2 = \frac{4\pi e^2}{m} \int_{-\infty}^{\infty} \frac{f_0' dv}{v-u}, \qquad (10a)$$

where m, e are the mass and charge of an electron. It can be seen that (10) and (10a) differ in sign.

3. Stability Problem

Equation (10) defines growing (Im u > 0) and stationary (Im u = 0) waves. The dispersion relation (10) does not describe damped waves because they cannot exist at an arbitrary instant of time as is assumed in substitutional analysis. Damped waves depend strongly on the initial conditions and the appropriate dispersion equation has another form [3].

If therefore there exists a solution of (10) in the upper half-plane, the system will be unstable. Consequently, the absence of roots of (10) is a criterion for stability. An analogous situation arises in the case of a plasma. The stability of a plasma was studied by Noerdlinger [4].

Noerdlinger proved that the plasma is stable if and only if

$$U(p) = P \int_{-\infty}^{\infty} \frac{f_0' dv}{v-p} < 0 \qquad (11)$$

at every minimum of f_0. Here p is a point where f_0 assumes a minimum value, and the symbol P indicates the principal value.

In proving the theorem in [4] use was made of the similarity of an integral in (10a) to the potential of an electrically-charged filament. Applying integration by parts, the integral $\int_{-\infty}^{\infty} \frac{f_0' dv}{v-u}$ can be expressed in the form

$$-\int_{-\infty}^{\infty} \ln|v-u| f_0'' dv \equiv W(u) = U(u) + iV(u). \qquad (12)$$

From (10a) and (12) it is clear that the plasma is unstable when there is a point u for which

$$U(u) > 0, \quad V(u) = 0, \quad \operatorname{Im} U > 0 \qquad (13)$$

for a real value of k.

It follows from (13) that the plasma is stable if and only if U(u) < 0 for all points on the curve V(u) = 0 in the open upper half of the u-plane.

The problem thus ends in a study of the sign of V(u) on the curves V(u) = 0. Noerdlinger carried this out with the aid of an electrostatic analogy and the theory of the complex potential. W(u) determines a plane vector field. In the electrostatic case, the vector of the field $\mathbf{E} = -\nabla U$.

Near the real axis of the u-plane we have

$$W(u) \longrightarrow P \int_{-\infty}^{\infty} \frac{f_0' dv}{v - \operatorname{Re} u} + i\pi f_0'(\operatorname{Re} u). \qquad (14)$$

Since by (10a), $\pi f_0'(\operatorname{Re} u) = V = 0$, at the points Re u = p, where V = 0 intersects the real axis, f_0 assumes an extreme value. Insofar as the charges are situated on the real axis, the potential U is also extremal at points where the curve V = 0 intersects the real axis, and has a maximum at points where the lines of force V = 0 leave the real axis.

It follows from the Cauchy-Riemann conditions that on the real axis the normal component of the electric field is $E_\perp = -\frac{\partial U}{\partial(\operatorname{Im} u)} = \frac{\partial V}{\partial(\operatorname{Re} u)} = \pi f_0''$.

At points where the lines of force V = 0 depart from the real axis $\partial U/\partial(\operatorname{Im} u) < 0$, since the potential decreases along the lines of force. Consequently, f_0 assumes a maximum value at point p. It can also be proved that U is sign invariant on each point of the curve V = 0 in the upper half-plane u, if pU(u) < 0 [4]. This proves the theorem.

It should be noted that if f_0 does not have a minimum but a maximum (for example, one-peak distributions), then integration of U(p) by parts under the condition $f_0'(p) = 0$ yields

$$U(p) = \int_{-\infty}^{\infty} \frac{f_0(v) - f_0(p)}{(v-p)^2} dv. \qquad (15)$$

Then it is clear that U(p) < 0 as the integral of a negative function. Thus, the analogous distributions in a plasma are stable.

It is easy to see that a gravitating system is stable when (13) is satisfied for all u.

The considered locally homogeneous segment of a gravitating system is therefore stable if and only if

$$U(p) = P\int_{-\infty}^{\infty} \frac{f_0' dv}{v-p} > 0 \qquad (16)$$

at all roots of $f_0' = 0$, where p is a root of $f_0' = 0$.

One can generalize (15) in a similar way. Hence, in the case of gravitation, the system is locally unstable when f_0 has at least one maximum. Obviously, one encounters exactly such systems in nature. Therefore actual gravitating systems (galaxies, star clusters, etc.) are locally unstable.

Stability in the sense formulated above means the absence of any wavelengths with respect to which the system is unstable. If a system is unstable in this sense, there is a critical Jeans' wavelength that determines the boundaries of stability of the system [5]. In the homogeneous case, λ_{cr} can be determined by the formula [5]

$$\left(\frac{2\pi}{\lambda_{cr}}\right)^2 = k_{cr}^2 = -\alpha P\int_{-\infty}^{\infty} \frac{f_0'(v)}{v-u_m} dv, \qquad (16a)$$

where u_m is the principal maximum of f_0. For wavelength $\lambda < \lambda_{cr}$ the waves are damped. This is Landau damping [3].

We have not taken account of the rotation, which can stabilize the system. Thus, strictly speaking, when the theory is applied to rotating systems it only describes waves along the axis of rotation.

It could also be pointed out that the present investigation is not complete in the sense that the type of instability arising remains undetermined. It is still necessary to clarify whether the stability is of the drifting or absolute type. However, this question will not be touched upon in this article.

4. Particular Cases of Gravitating Systems

1. "Cold" Systems of Gravitating Particles. When frequent collisions take place in a gravitating system, it can be described by hydrodynamic equations. In this case it is well known that the dispersion equation for small oscillations of a homogeneous medium at rest has the form

$$u^2 = \frac{\omega^2}{k^2} = c_T{}^2 - \frac{\alpha \cdot N}{k^2}, \qquad (17)$$

where c_T is the velocity of sound.

The physical meaning of (17) is that the stability or instability of the system is determined by the balance of pressure (or of temperature) that tends to decrease the density disturbances, and the gravitational attraction, which acts in the opposite direction. If the pressure (temperature) is sufficiently high, it hinders the gravitational compression and the system is stable. Insofar as the attraction that produces instability can be determined by the size of the perturbation, it is clear that this size is decisive. Thus, it follows from (17) that the system is stable when $c_T^2 > \alpha N/k^2$, i.e., at high temperatures, and the system is unstable when $c_T^2 < \alpha N/k^2$. For example, when $c_T = 0$ (cold gas),

$$\omega^2 = -\alpha N. \qquad (18)$$

From (17) one finds the critical size of the perturbed region λ_{cr} (Jeans' critical wavelength); when $\lambda > \lambda_{cr}$ the system is always unstable.

In a cold medium ($u_T = 0$) the particles rest in a state of equilibrium. Thus, a cold gas has the same properties as a cold collisionless system of gravitating particles. Consequently, it can be expected that the dispersion relation for such a system will be identical with (18). In fact, $f_0 = N\delta(v)$ in this case, and then (10) yields (18).

2. "Cold" Streams. We will consider two streams of equal density N/2 traveling in opposite directions with the velocities $\pm v_1$. In this case $f_0 = (N/2)\{\delta(v-v_1) + \delta(v+v_1)\}$. Then from (10) we obtain the dispersion equation

$$k^2 = -\frac{\alpha N}{2}\left[\frac{1}{(v_1-u)^2} + \frac{1}{(v_1+u)^2}\right]. \qquad (19)$$

When (19) is solved graphically [6], it is easy to verify that all four roots are complex and that they appear as conjugate pairs. Thus, in two cases $\mathrm{Im}\,\omega > 0$ and the system is always unstable in contrast to the case of a plasma.

When one of the two cold streams is not present, (19) assumes the form

$$k^2 = -\frac{\alpha N}{2(v_1-u)^2}. \qquad (19a)$$

whence

$$u = \frac{\omega}{k} = v_1 \pm i\sqrt{\frac{\alpha N}{2k^2}}. \qquad (20)$$

It follows from (20) that the instability of such a system has a pulsating character and is not aperiodic as in the case of cold media at rest. The resulting fluctuations have been explained as originating in asymmetry. In fact, if the perturbation of the potential moves together with the streams, the particles that move toward the perturbation and those that overtake it will have different velocities. The perturbation of the gravitational potential is equivalent to a gravitational well. The particles that travel from the potential barrier toward the center in the direction of the flow of the stream will have a lower speed than particles traveling in the opposite direction. For example, when the velocity of the stream is high, the particles of the first class will be almost at rest. The particles of the second class, on the other hand, will have a high speed and, when reflected from the potential barrier, they execute frequent oscillations. Thus, the density of the particles at each point in the well fluctuates. The instability is caused by the capture in the well of additional particles.

In the case of two cold streams, pulsations arise when $\lambda < \lambda_1$. In fact, from (19) we have

$$u^2 = v_1^2 - \frac{\alpha N}{2k^2}\left(1 \pm \sqrt{1 - 8\frac{k^2 v_1^2}{\alpha N}}\right). \qquad (21)$$

The sign before the square root in (21) is found from the condition $\lim_{v_1 \to 0} u^2 = -\alpha N/k^2$, i.e., it is positive. Now it follows from (21) that pulsations will occur when

$$1 - 8\frac{k^2 v_1^2}{\alpha N} < 0, \qquad (22)$$

i.e., when

$$\lambda < \lambda_1 = \pi v_1 \sqrt{\frac{32}{\alpha N}}. \qquad (23)$$

This is obviously connected with the fact that for sufficiently large dimensions the perturbed region is asymmetrically smoothed.

3. System with Maxwellian Distribution. In this case

$$f_0 = \frac{N}{\sqrt{\pi}\, u_T} e^{-v^2/u_T^2} \qquad (24)$$

where $u_T^2 = 2\theta/M$.

By making use of the results in [7,8], we obtain from (10)

$$k^2 = \frac{2\alpha N}{u_T^2}\{1 + i\sqrt{\pi}\,\beta e^{-\beta^2}[1 + \varphi(i\beta)]\}, \qquad (25)$$

where $\beta = u/u_T$, and φ is the error integral.

Formula (25) determines the type of waves that arise in the given system. In view of the difficulty in exactly solving the transcendental equation (25), we will study its asymptotic behavior. When $|\beta| \gg 1$ (low temperatures), (25) yields

$$\frac{k^2 u_T^2}{\alpha N} \approx -\frac{1}{\beta^2}\left(1 + \frac{3}{2\beta^2} + \frac{15}{4\beta^4}\right). \qquad (26)$$

Limiting ourselves to terms of the order β^{-2} compared with unity, we obtain (18) in the following expression:

$$\beta^2 = -\frac{\alpha N}{2k^2 u_T^2} - \sqrt{\frac{\alpha^2 N^2}{4k^4 u_T^4} - \frac{3}{2}\cdot\frac{\alpha N}{k^2 u_T^2}}. \qquad (27)$$

The sign in front of the square root has been chosen in order that (27) tend to (18) as $u_T \to 0$.

Expression (27) infers the presence of a pulsating instability when

$$\lambda < \lambda_1 = \pi u_T \sqrt{\frac{24}{\alpha N}}. \qquad (28)$$

The way in which the instability arises is exactly the same as in the case of interpenetrating cold streams, because the Maxwellian distribution on the average is equivalent to streams moving with the velocities $\pm u_T$.

The inverse value of the increment (the instability time), for example, when $\lambda > \lambda_1$, according to order of magnitude is equal to

$$\tau = \frac{2\pi}{\mathrm{Im}\,\omega}, \qquad (29)$$

$$\mathrm{Im}\,\omega \approx \sqrt{\frac{\alpha N}{2}}\sqrt{1 + \sqrt{1 - 6\frac{k^2 u_T^2}{\alpha N}}}. \qquad (29a)$$

4. Interpenetrating Streams with Maxwellian Velocity Distributions. Here f_0 has the form

$$f_0 = \frac{N}{2\sqrt{\pi}\, u_T}\left(\exp\left[-\frac{(v - v_1)^2}{u_T^2}\right] + \exp\left[-\frac{(v + v_1)^2}{u_T^2}\right]\right). \qquad (30)$$

In this case the dispersion equation (10) becomes

$$k^2 = \frac{\alpha N}{u_T^2}\{2 \pm i\sqrt{\pi}(\beta - \eta)$$

$$\times \exp[-(\beta - \eta)^2][1 \pm \varphi[i(\beta - \eta)]] \pm$$

$$\pm i\sqrt{\pi}(\beta+\eta)\exp[-(\beta+\eta)^2][1\pm\varphi[i(\beta+\eta)]]\}, \quad (31)$$

where $\eta = v_1/u_T$, and the signs (±) correspond to waves traveling along the x axis in the positive and negative directions. When $|\beta| \gg 1$, one obtains the so-called "cold" approximation from (31), i.e., limiting oneself to terms of the order $(\beta \pm \eta)^{-2}$:

$$\beta^2 \approx -\left(\frac{\alpha N}{2k^2 u_T^2} - \eta^2\right) - \sqrt{\frac{\alpha N}{2k^2 u_T^2}\left(\frac{\alpha N}{2k^2 u_T^2} - 4\eta^2\right)}. \quad (32)$$

Here also the instability acquires pulsating character when $\lambda < \lambda_1$, which satisfies (23). The time to develop instability is determined by formula (29), where $\mathrm{Im}\,\omega$ is found from (32).

5. Application to the Galaxy. By using the theory developed above, one could consider the possible types of waves, the instability times, and other related questions for various stellar systems. It is well known that in the majority of systems that have been studied the distribution of the peculiar velocities in different directions can be well described by either the Schwarzschild law, or, in terms of interpenetrating stellar streams, by the Kapteyn-Eddington law. Since both models agree equally well with the observations, either model can be adopted. Following [2], we will adopt the stream model. Thanks to methods developed by Eddington [9, 10], the parameters (u_T, v_1) of the streams can easily be determined from observations. Thus, by using the results of Sections 3 and 4, it is possible to study the types of waves in the locally homogeneous regions of different stellar systems. As was emphasized above, the extension of the results obtained to systems is on the whole incorrect, since according to the initial assumptions consideration has been given to small regions of the system, only, where the gravitational potential can be assumed to be almost constant. We will now consider the Galaxy in the neighborhood of the sun.

In the direction Z of the axis of rotation, it is apparent that $f_0(v_z)$ is a Maxwellian function with u_T = 18 km/sec [11]. In this direction the dimension of the locally homogeneous region is $l \approx$ 0.38 kpc [12]. The density is $\rho = NM \approx 3.5 \cdot 10^{-24}$ g/cm^3 [2]. The critical wavelength on the basis of these values of u_T and ρ turns out to be λ_{cr} = 1.25 kpc. According to (28), the instability assumes pulsating character starting with $\lambda_1 = (12)^{1/2}{}_{cr}$.

It is known [2, 13] that in the galactic plane the function f_0 can be well approximated by formula (30). Small differences in the magnitudes of v_1 and N in both streams will be neglected. According to [2], v_1 = 21 km/sec, u_T = 25 km/sec, $\rho \approx 3.5 \cdot 10^{-24}$ g cm^3. Then, λ_{cr} = 5.32 kpc, λ_1 = 8 kpc according to the approximation (23), and $l \approx 1.4$ kpc [14].

For a rough estimate of τ, use can be made of (29). Taking $\mathrm{Im}\,\omega \approx (\alpha N)^{1/2}$, we obtain $\tau \approx 2 \cdot 10^8$ years. This value of τ is of the order of the period of rotation of the Galaxy in the neighborhood of the sun but is significantly shorter than relaxation time due to pairs of approaching stars.

It is clear that, in the galactic plane and in the direction of the Z axis, λ_{cr} and λ_1 are larger than l. For this reason, when the results of the homogeneous theory are applied to the Galaxy they have a formal character since the homogeneous theory is inapplicable to regions of systems with a large value of l. Consequently it is impossible to say anything about the stability of the Galaxy on the basis of the homogeneous theory; inhomogeneous treatment is necessary.

The second part of this work will study waves in inhomogeneous entities. A general discussion of the results relating to locally homogeneous regions and to systems as a whole will be given in this journal.

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